Published by Institute of Physics Publishing for SISSA

Received: *May 16, 2007*
Accepted: *July 6, 2007*
Published: *July 17, 2007*

Marginal solutions for the superstring

Theodore G. Erler

Harish-Chandra Research Institute,
Chhatnag Road, Jhunsi, Allahabad 211019, India
E-mail: terler@mri.ernet.in

Abstract: We construct a class of analytic solutions of WZW-type open superstring field theory describing marginal deformations of a reference D-brane background. The deformations we consider are generated by on-shell vertex operators with vanishing operator products. The superstring solution exhibits an intriguing duality with the corresponding marginal solution of the *bosonic* string. In particular, the superstring problem is "dual" to the problem of re-expressing the bosonic marginal solution in pure gauge form. This represents the first nonsingular analytic solution of open superstring field theory.

Keywords: String Field Theory, Tachyon Condensation.

Contents

JHEP07(2007)050

1. Introduction

Following the breakthrough analytic solution of Schnabl [1], our analytic understanding of open string field theory (OSFT) has seen remarkable progress [2 – 9]. So far most work has focused on the open bosonic string, but clearly it is also important to consider the superstring. This is not just because superstrings are ultimately the theory of interest, but because there are important physical questions, especially the holographic encryption of closed string physics in OSFT, which may be difficult to decipher in the bosonic case [10].

Ideally, the first goal should be to find an analytic solution of superstring field theory[1] on a non-BPS brane describing the endpoint of tachyon condensation, i.e. the closed string vacuum. However, the construction of this solution will likely be subtle — indeed, Schnabl's solution for the bosonic vacuum is very close to being pure gauge [1, 2]. Thus, it may be useful to consider a simpler problem first: constructing solutions describing *marginal deformations* of a (non)BPS D-brane. Marginal deformations correspond to a one-parameter family of open string backgrounds obtained by adding a conformal boundary interaction to the worldsheet action — for example, turning on a Wilson line on a brane by adding the boundary term $A_\mu \int_{\partial\Sigma} dt \partial X^\mu(t)$ to the worldsheet action. Such backgrounds were studied numerically for the bosonic string in ref. [12] and for the superstring in ref. [13]. Recently, Schnabl [14] and Kiermaier *et al*[15] found *analytic* solutions for marginal deformations in bosonic OSFT.[2] The solutions bear striking resemblance to Schnabl's vacuum solution, but are simpler in the sense that they are manifestly nontrivial and can be constructed systematically with a judicious choice of gauge.

[1]In this paper we will work with the Berkovits WZW-type superstring field theory [11].

[2]For previous analytic studies of marginal solutions in bosonic and super OSFT, see refs. [16, 17].

In this note, we construct solutions of super OSFT describing marginal deformations generated by on-shell vertex operators with *vanishing* operator products (in either the 0 or -1 picture). As was found in ref. [14, 15] such deformations are technically simpler since they allow for solutions in Schnabl's gauge, $\mathcal{B}_0\Phi = 0$ — though probably more general marginal solutions can be obtained once the analogous problem is understood for the bosonic string, either by adding counterterms as described in ref. [15] or by employing a "pseudo-Schnabl gauge" as suggested in ref. [14]. The superstring solution exhibits a remarkable duality with its bosonic counterpart: it formally represents a re-expression of the bosonic solution in pure gauge form. It would be very interesting if this duality generalized to other solutions.[3]

This paper is organized as follows. In section 2 we briefly review the bosonic marginal solution in the split string formalism [2, 8, 20], which we will prove convenient for many computations. In section 3 we consider the superstring, motivating the solution as analogous to constructing an explicit pure gauge form for the bosonic marginal solution. This strategy quickly gives a very simple expression for the complete analytic solution of super OSFT. In section 4 we consider the dual problem: finding a pure gauge expression for the bosonic marginal deformation describing a constant, light-like gauge field on a non-compact brane. Though quite analogous to the superstring, this problem is slightly more complex. Nevertheless we are able to find an analytic solution. We end with some conclusions.

While this note was in preparation, we learned of the independent solution by Yuji Okawa [21]. His paper should appear concurrently.

2. Bosonic solution

Let us begin by reviewing the bosonic marginal solution [14, 15] in the language of the split string formalism [2, 8, 20], which is a useful shorthand for many calculations. The first step in this approach is to find a subalgebra of the open string star algebra, closed under the action of the BRST operator, in which we hope to find an analytic solution. For the bosonic marginal solution the subalgebra is generated by three string fields K, B and J:

$$\begin{aligned} K &= \text{Grassmann even, gh\# = 0} \\ B &= \text{Grassmann odd, gh\# = -1} \\ J &= \text{Grassmann odd, gh\# = 1} \end{aligned} \tag{2.1}$$

satisfying the identities,

$$[K, B] = 0 \qquad B^2 = J^2 = 0 \tag{2.2}$$

and

$$dK = 0 \quad dJ = 0 \quad dB = K \tag{2.3}$$

[3] A related observation was made by Ohmori in the context of vacuum superstring field theory [18], and also plays an important role in the construction of heterotic string field theory [19].

where $d = Q_B$ is the BRST operator and the products above are open string star products (we will mostly omit the $*$ in this paper). The relevant explicit definitions of K, B, J are,[4]

$$\begin{aligned} K &= -\frac{\pi}{2}(K_1)_L|I\rangle \qquad K_1 = L_1 + L_{-1} \\ B &= -\frac{\pi}{2}(B_1)_L|I\rangle \qquad B_1 = b_1 + b_{-1} \\ J &= J(1)|I\rangle \end{aligned} \tag{2.4}$$

where $|I\rangle$ is the identity string field and the subscript L denotes taking the left half of the corresponding charge.[5] The operator $J(z)$ is a dimension zero primary generating the marginal trajectory. It takes the form,

$$J(z) = c\mathcal{O}(z) \tag{2.5}$$

where $\mathcal{O}$ is a dimension one matter primary with *nonsingular* OPE with itself. This is crucial for guaranteeing that the square of the field J vanishes, as in eq. (2.2). With these preliminaries, the marginal solution for the bosonic string is:

$$\Psi = \lambda F J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F \tag{2.6}$$

where λ parameterizes the marginal trajectory and $F = e^{K/2} = \Omega^{1/2}$ is the square root of the $\mathrm{SL}(2,\mathbb{R})$ vacuum (a wedge state). To linear order in λ the solution is,

$$\Psi = \lambda F J F + \cdots = \lambda J(0)|\Omega\rangle + \cdots \tag{2.7}$$

which is the nontrivial element of the BRST cohomology generating the marginal trajectory.

Let us prove that eq. 2.6 satisfies the equations of motion. Using the identities eqs. (2.2), (2.3),

$$\begin{aligned} d\Psi &= -\lambda F J d\left(\frac{1}{1 - \lambda B \frac{F^2-1}{K} J}\right) F \\ &= -\lambda F J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} d\left(\lambda B \frac{F^2-1}{K} J\right) \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F \\ &= -\lambda^2 F J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} (F^2-1) J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F \end{aligned} \tag{2.8}$$

Notice the $(F^2 - 1)J$ factor in the middle. Since $J^2 = 0$, the $\ldots - 1)J$ term vanishes when multiplied with the Js to the left — thus the necessity of marginal operators with nonsingular OPE. This leaves,

$$d\Psi = -\lambda^2 F J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F^2 J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F = -\Psi^2 \tag{2.9}$$

[4] We may generalize the construction by considering other projector frames [4, 7, 8] or by allowing the field F in eq. (2.6) to be an arbitrary function of K [2, 9]. Such generalizations do not add much to the current discussion so we will stick with the definitions presented here.

[5] "Left" means integrating the current counter-clockwise on the positive half of the unit circle. This convention differs by a sign from ref. [8] but agrees with ref. [4].

i.e. the bosonic equations of motion are satisfied.

The solution has a power series expansion in λ:

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \Psi_n \tag{2.10}$$

where,

$$\Psi_n = FJ\left(B\frac{F^2-1}{K}J\right)^{n-1} F \tag{2.11}$$

To make contact with the expressions of refs. [14, 15], note the relation,

$$\frac{F^2-1}{K} = \int_0^1 dt\Omega^t \tag{2.12}$$

To prove this, recall $\Omega^t = e^{tK}$ and calculate,[6]

$$K\int_0^1 dt\Omega^t = \int_0^1 dt\frac{d}{dt}e^{tK} = e^K - 1 = F^2 - 1 \tag{2.13}$$

Using this and the mapping between the split string notation and conformal field theory described in ref. [8], the Ψ_ns can be written as CFT correlators on the cylinder:

$$\begin{aligned}\langle\Psi_n,\chi\rangle = \int_0^1 dt_1 \ldots \int_0^1 dt_{n-1}\langle J(t_{n-1}+\cdots+t_1+1)B\ldots J(t_1+1)\times \\ \times BJ(1)f_{\mathcal{S}}\circ\chi(0)\rangle_{C_{t_{n-1}+\cdots+t_1+2}}\end{aligned} \tag{2.14}$$

where $f_{\mathcal{S}}(z) = \frac{2}{\pi}\tan^{-1} z$ is the sliver conformal map, and in this context B is the insertion $\int_{i\infty}^{-i\infty}\frac{dz}{2\pi i}b(z)$ to be integrated parallel to the axis of the cylinder in between the J insertions on either side. This matches the expressions found in refs. [14, 15].

In passing, we mention that this solution was originally constructed systematically by using the equations of motion to recursively determine the Ψ_ns in Schnabl gauge. If desired, it is also possible to perform such calculations in split string language; we offer some sample calculations in appendix A.

3. Superstring solution

Let us now consider the superstring. The marginal deformation is generated by a -1 picture vertex operator,

$$e^{-\phi}c\mathcal{O}(z) \tag{3.1}$$

where $\mathcal{O}(z)$ is a dimension $\frac{1}{2}$ superconformal matter primary. We will use Berkovits's WZW-type superstring field theory [11],[7] in which case the string field is given by multiplying the -1 picture vertex operator by the ξ ghost:

$$X(z) = \xi e^{-\phi}c\mathcal{O}(z) \tag{3.2}$$

[6]Note that, in general, the inverse of K is not well defined. However, when operating on F^2-1 it is. This is why we cannot simply use F^2/K in the solution in place of $\frac{F^2-1}{K}$, which would naively give a solution even for marginal operators with singular OPEs.

[7]See refs. [22–24] for nice reviews.

This corresponds to a solution of the linearized Berkovits equations of motion,

$$\eta_0 Q_B \left(\lambda X(0)|\Omega\rangle\right) = 0 \tag{3.3}$$

since η_0 eats the ξ and the -1 picture vertex operator is in the BRST cohomology. We will also find it useful to consider the 0 picture vertex operator,

$$J(z) = Q_B \cdot X(z) = cG_{-1/2} \cdot \mathcal{O}(z) + \eta e^{\phi}\mathcal{O}(z) \tag{3.4}$$

A complimentary way of seeing the linearized equations of motion are satisfied is to note that $J(z)$ is in the small Hilbert space. As with the bosonic string, it is very helpful to assume that $X(z)$ and $J(z)$ have vanishing OPEs:

$$\lim_{z\to w} J(z)X(w) = \lim_{z\to w} J(z)J(w) = \lim_{z\to w} X(z)X(w) = 0 \tag{3.5}$$

We mention two examples of such deformations. The simplest is the light-like Wilson line $\mathcal{O}(z) = \psi^+(z)$ ($\alpha' = 1$), where

$$\begin{aligned} X(z) &= \xi e^{-\phi} c\psi^+(z) \\ J(z) &= i\sqrt{2} c\partial X^+(z) + \eta e^{\phi}\psi^+(z) \end{aligned} \tag{3.6}$$

There is also a "rolling tachyon" marginal deformation [25] $\mathcal{O}(z) = \sigma_1 e^{X^0/\sqrt{2}}(z)$ on a non-BPS brane. The corresponding vertex operators are,

$$\begin{aligned} X(z) &= \sigma_1 \xi e^{-\phi} c e^{X^0/\sqrt{2}}(z) \\ J(z) &= i\sigma_2(-ic\psi^0 + \eta e^{\phi}) e^{X^0/\sqrt{2}}(z) \end{aligned} \tag{3.7}$$

The Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are "internal" Chan-Paton factors [26, 27], necessary to accommodate non-BPS GSO$(-)$ states into the Berkovits framework. Though we will not write it explicitly, in this context it is important to remember that the BRST operator and the eta zero mode are carrying a factor of σ_3 (thus the presence $i\sigma_2 = \sigma_3\sigma_1$ in the above expression for J). We mention that both $X(0)|\Omega\rangle$ and $J(0)|\Omega\rangle$ are in Schnabl gauge and annihilated by $\mathcal{L}_0$.

Let us describe the subalgebra relevant for finding the marginal solution. It consists of the products of four string fields, K, B, X, J:

$$\begin{aligned} K &= \text{Grassmann even},\ \text{gh}\# = 0 \\ B &= \text{Grassmann odd},\ \text{gh}\# = -1 \\ X &= \text{Grassmann even},\ \text{gh}\# = 0 \\ J &= \text{Grassmann odd},\ \text{gh}\# = 1 \end{aligned} \tag{3.8}$$

All four of these have vanishing picture number. K and B are the same fields encountered earlier in eq. (2.4); X and J are defined,

$$X = X(1)|I\rangle \qquad J = J(1)|I\rangle \tag{3.9}$$

with $X(z), J(z)$ as in eqs. (3.2), (3.4). We have the identities,

$$[K,B]=0 \quad B^2=0 \quad X^2=J^2=XJ=JX=0 \tag{3.10}$$

where the third set follows because the corresponding vertex operators have vanishing OPEs. The algebra is closed under the action of the BRST operator:

$$\begin{aligned} dB &= K \qquad dK=0 \\ dX &= J \qquad\ \, dJ=0 \end{aligned} \tag{3.11}$$

Note that the eta zero mode $\bar{d}\equiv\eta_0$ annihilates K, B and J,

$$\bar{d}K=\bar{d}B=\bar{d}J=0 \tag{3.12}$$

since they live in the small Hilbert space. However, it does *not* annihilate X, and the algebra is not closed under $\bar{d}$. Though it is not *a priori* obvious that the K, B, X, J algebra is rich enough to encapsulate the marginal solution, we will quickly see that it is.[8]

We seek a one parameter family of solutions of the super OSFT equations of motion,

$$\bar{d}\left(e^{-\Phi}de^{\Phi}\right)=0 \tag{3.13}$$

where Φ is a Grassmann even, ghost and picture number zero string field which to linear order in the marginal parameter takes the form,

$$\Phi=\lambda FXF+\cdots \tag{3.14}$$

There are many strategies one could take to solve this equation, but before describing our particular approach it is worth mentioning the "obvious" method: fixing Φ in Schnabl gauge and attempting a perturbative solution, as in refs. [14, 15]:

$$\Phi=\sum_{n=1}^{\infty}\lambda^n\Phi_n \qquad \Phi_1=FXF \tag{3.15}$$

At second order,[9] the Schnabl gauge solution is actually fairly simple:

$$\Phi_2=\frac{1}{2!}\left[FXB\frac{F^2-1}{K}JF+FJB\frac{F^2-1}{K}XF\right] \tag{3.16}$$

[8]For GSO($-$) deformations the above discussion is subtly modified. In particular, X must be Grassmann *odd* while J is Grassmann *even*. Still, effectively the Grassmann assignments eq. (3.8) remain valid since X, J carry internal Chan-Paton factors which anticommute with the σ_3s carried by Q_B, η_0. If we like, we can take the effective Grassmann parity to be the "bare" Grassmann parity plus the number of σ_1s. Also, note that the field B should implicitly carry a factor of σ_3 in order for $dB=K$. Since the solution has ghost number zero, J and B always appear multiplied, so we could absorb B's Chan-Paton factor into J, which amounts to replacing the $i\sigma_2$ in eq. (3.7) by σ_1. Thus the field has the expected decomposition $\Phi=\Phi^{\mathrm{GSO}(+)}+\sigma_1\Phi^{\mathrm{GSO}(-)}$.

[9]Explicitly, if we plug eq. (3.15) into the equations of motion, we find a recursive set of equations of the form $\bar{d}d\Phi_n=\bar{d}\mathcal{F}_{n-1}[\Phi]$, where $\mathcal{F}_{n-1}[\Phi]$ depends on $\Phi_1,\ldots,\Phi_{n-1}$. The Schnabl gauge solution is obtained by writing $\Phi_n=\frac{\mathcal{B}_0}{\mathcal{L}_0}\mathcal{F}_{n-1}[\Phi]$.

JHEP07(2007)050

and seems quite similar to the bosonic solution. At third order, however, we found an extremely complicated expression (though still within the K, B, X, J subalgebra). It seems doubtful that a closed form solution for Φ in Schnabl gauge can be obtained.

Since the Schnabl gauge construction appears complicated, we are lead to consider another approach. To motivate our particular strategy, we make two observations: First, the combination $e^{-\Phi}de^{\Phi}$ which enters the superstring equations of motion *also* happens to be a pure gauge configuration from the perspective of bosonic OSFT. Second, there is a basic similarity between the K, B, J algebra for the bosonic marginal solution and the K, B, J, X algebra for the superstring. The main difference of course is the presence of X for the superstring, whose BRST variation gives J. If such a field were present for the bosonic string, the bosonic marginal solution would be pure gauge because J would be trivial in the BRST cohomology. With this motivation, we are lead to consider the equation

$$e^{-\Phi}de^{\Phi} = \lambda FJ\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \tag{3.17}$$

From the bosonic string perspective, this equation represents an expression of the bosonic marginal solution in a form which is pure gauge. From the superstring perspective, this is a partially gauge fixed form of the equations of motion, since the expression on the right hand side is in the small Hilbert space.

Let us now solve this equation. It will turn out to be simpler to solve for the group element $g = e^{\Phi}$; we make a perturbative ansatz,

$$g = e^{\Phi} = 1 + \sum_{n=1}^{\infty}\lambda^n g_n \qquad\qquad g_1 = \Phi_1 = FXF \tag{3.18}$$

Expanding out eq. (3.17) to second order gives,

$$\begin{aligned} dg_2 &= FJB\frac{F^2-1}{K}JF + g_1 dg_1 \\ &= FJB\frac{F^2-1}{K}JF + FXF^2JF \end{aligned} \tag{3.19}$$

As it turns out, this equation is solved by the second order Schnabl gauge solution eq. (3.16):

$$g_2 = \Phi_2 + \frac{1}{2}\Phi_1^2 = \frac{1}{2!}\left[FXB\frac{F^2-1}{K}JF + FJB\frac{F^2-1}{K}XF + FXF^2XF\right] \tag{3.20}$$

but there is a simpler solution:

$$g_2 = FXB\frac{F^2-1}{K}JF \tag{3.21}$$

Using this form of g_2 we can proceed to third order — remarkably, the solution is practically just as simple:

$$g_3 = FX\left(B\frac{F^2-1}{K}J\right)^2 F \tag{3.22}$$

This leads to an ansatz for the full solution:

$$e^{\Phi} = 1 + \lambda F X \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \tag{3.23}$$

To check this, calculate:

$$\begin{aligned}
de^{\Phi} &= \lambda F J \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda F X d\left(\frac{1}{1-\lambda B\frac{F^2-1}{K}J}\right)F \\
&= \lambda F J \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda F X \frac{1}{1-\lambda B\frac{F^2-1}{K}J} d\left(\lambda B\frac{F^2-1}{K}J\right)\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\
&= \lambda F J \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda^2 F X \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F^2 J\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\
&= \left(1 + \lambda F X \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F\right)\lambda F J \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\
&= e^{\Phi}\lambda F J \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F
\end{aligned} \tag{3.24}$$

Therefore, eq. (3.23) is indeed a complete solution to the super OSFT equations of motion! Note, however, that it is not quite a solution to the pure gauge problem of the bosonic string. In particular, in step three we needed to assume $XJ = 0$ — something we would not expect to hold in the bosonic context. We will give the solution to the bosonic problem in the next section.

Let us make a few comments about this solution. First, though the string field Φ itself is not in Schnabl gauge, the nontrivial part of the group element e^{Φ} *is* — this is not difficult to see, but we offer one explanation in appendix A. The second comment is related to the string field reality condition. In super OSFT, the natural reality condition is that Φ should be "imaginary" in the following sense:

$$\langle \Phi, \chi\rangle = -\langle \Phi|\chi\rangle \tag{3.25}$$

where $\langle\Phi|$ is the Hermitian dual of $|\Phi\rangle$ and χ is any test state. Alternatively, we can write this,

$$\Phi^{\dagger} = -\Phi \tag{3.26}$$

where $\dagger$ is an anti-involution on the star algebra, formally completely analogous to Hermitian conjugation of operators.[10] With this reality condition, the group element should be

[10] Given a state Ψ, we define its conjugate $\Psi^{\dagger}$,

$$\langle \Psi^{\dagger}, \chi\rangle = \langle \Psi|\chi\rangle$$

If $|\Psi\rangle$ carries Chan-Paton indices, then to get the Hermitian dual $\langle\Psi|$ we should also transpose the indices. We have the notable properties,

$$(\Psi^{\dagger})^{\dagger} = \Psi \qquad (\Psi\Phi)^{\dagger} = \Phi^{\dagger}\Psi^{\dagger} \qquad (d\Psi)^{\dagger} = (-1)^{\Psi+1}d(\Psi^{\dagger})$$

The third equation is true even when d carries an internal Chan-Paton factor σ_3, provided that $(-1)^{\Psi}$ is taken to be the "effective" Grassmann parity. We have to be a little careful with this conjugation for vertex operators of non-integer conformal weight, though such subtleties play no role in this paper. For a more detailed discussion of the reality condition in open superstring field theory, see ref. [28].

unitary:

$$g^\dagger = g^{-1}$$

Using,

$$K^\dagger = K \quad B^\dagger = B \quad J^\dagger = J \quad X^\dagger = -X \tag{3.27}$$

it is not difficult to see that the analytic solution e^Φ is *not* unitary. By contrast, the Schnabl gauge construction automatically gives an imaginary Φ and unitary e^Φ. However, upon further analysis we have found a way to construct a unitary e^Φ by gauge transformation of eq. (3.23). Details are explained in an added appendix B.

Let us take the opportunity to express the solution in a few other forms which may be more convenient for explicit computations. Following the usual prescription we may express the g_ns as correlation functions on the cylinder:

$$\begin{aligned}\langle g_n, \chi\rangle &= \int_0^1 dt_1 \ldots \int_0^1 dt_{n-1} \langle X(t_{n-1} + \cdots + t_1 + 1) \times \\ &\qquad \times BJ(t_{n-2} + \cdots + t_1 + 1) \ldots BJ(1)\; f_{\mathcal{S}} \circ \chi(0)\rangle_{C_{\sum t_i + 2}} \quad (3.28)\\ &= (-1)^n \int_0^1 dt_1 \ldots \int_0^1 dt_{n-1} \langle X(L+1)[\mathcal{O}'(\ell_{n-2}+1) \ldots \mathcal{O}'(\ell_1+1)] \times \\ &\qquad \times BJ(1)\; f_{\mathcal{S}} \circ \chi(0)\rangle_{C_{L+2}} \quad (3.29)\end{aligned}$$

In the second line we manipulated the multiple B insertions, simplifying the vertex operators and obtaining a single B insertion to the right; we introduced the length parameters [15]:

$$\ell_i = \sum_{k=1}^{i} t_k \qquad L = \ell_{n-1} \tag{3.30}$$

and defined $\mathcal{O}'(z) = G_{-\frac{1}{2}} \cdot \mathcal{O}(z)$. We may also express the solution in the operator formalism of Schnabl [1]:

$$\begin{aligned}|g_n\rangle &= \frac{(-1)^{n\mathcal{O}+1}}{2} \int_0^1 dt_1 \ldots \int_0^1 dt_{n-1} \hat{U}_{L+2}\, f_{\mathcal{S}}^{-1} \circ (\xi e^{-\phi}\mathcal{O}(L/2))\tilde{\mathcal{O}}'(y_{n-2}) \ldots \tilde{\mathcal{O}}'(y_1)\\ &\quad \times \Big(\tilde{\mathcal{O}}'\Big(-\frac{L}{2}\Big)\Big[\mathcal{B}^+ \tilde{c}\Big(\frac{L}{2}\Big)\tilde{c}\Big(-\frac{L}{2}\Big) - \tilde{c}\Big(\frac{L}{2}\Big) - \tilde{c}\Big(-\frac{L}{2}\Big)\Big]\\ &\quad + f_{\mathcal{S}}^{-1} \circ \Big(\eta e^{\phi}\mathcal{O}\Big(-\frac{L}{2}\Big)\Big)\Big[\mathcal{B}^+ \tilde{c}\Big(\frac{L}{2}\Big) + 1\Big]\Big)|\Omega\rangle\end{aligned} \tag{3.31}$$

where $y_i = \ell_i - L/2$ and [6] $\hat{U}_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0^*}\left(\frac{2}{r}\right)^{\mathcal{L}_0}$. Also we have used $f_{\mathcal{S}}^{-1}$ to define the tilde to hide some factors of $\frac{\pi}{2}$. The expression is somewhat more complicated than the bosonic solution since the vertex operator $J(z)$ has a piece without a c ghost, so in the bc CFT the solution has a component not proportional to Schnabl's ψ_n [1].

4. Pure gauge for bosonic solution

In the last section, we found a solution for the superstring by analogy with the pure gauge problem of the bosonic string; but we did not solve the latter. The scenario we have in mind is a constant, lightlike gauge field on a non-compact D-brane. Since there is no flux and no way to wind a Wilson loop, such a field configuration should be pure gauge. From the string field theory viewpoint, this is reflected by the fact that the marginal vertex operator becomes BRST trivial in the noncompact limit,

$$ic\partial X^+(z) = Q_B \cdot 2iX^+(z) \tag{4.1}$$

Of course, on a compact manifold the operator $X^+(z)$ is not globally defined so the marginal deformation is nontrivial.

Translating to split string language, we consider an algebra generated by four fields K, B, X, J, where K, B are defined as before and,

$$X = 2iX^+(1)|I\rangle \qquad J = ic\partial X^+(1)|I\rangle \tag{4.2}$$

These have the same Grassmann and ghost number assignments as eq. (3.8). We have the algebraic relations,

$$[K,B]=0 \qquad B^2=0 \qquad J^2=0 \qquad [X,J]=0 \tag{4.3}$$

Note the difference from the superstring case: the products of X with itself and with J, though well defined (the OPEs are nonsingular), are nonvanishing. However, we still have

$$\begin{aligned} dB &= K \qquad dK = 0 \\ dX &= J \qquad\ \ dJ = 0 \end{aligned} \tag{4.4}$$

with the second set implying that J is trivial in the BRST cohomology.

We now want to solve eq. (3.17) assuming this slightly more general set of algebraic relations. Playing around a little bit, the solution we found is,

$$e^\Lambda = 1 + \lambda F u_\lambda(X) \frac{1}{1-\lambda B\frac{F^2-1}{K}J} F \tag{4.5}$$

where,

$$u_\lambda(X) = \frac{e^{\lambda X}-1}{\lambda} \tag{4.6}$$

The relevant identity satisfied by this particular combination is,

$$du_\lambda = J(\lambda u_\lambda + 1) \tag{4.7}$$

Let us prove that this gives a pure gauge expression for the bosonic marginal solution:

$$\begin{aligned} de^\Lambda &= \lambda F du_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda F u_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J} d\left(\lambda B\frac{F^2-1}{K}J\right)\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\ &= \lambda F J(\lambda u_\lambda + 1)\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda^2 F u_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J}(F^2-1)J\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \end{aligned}$$

Now we come to the critical difference from the superstring. Note the $\ldots - 1)J$ piece in the middle of the second term. Before it vanished when multiplied by X, J to the left. This time it contributes because $XJ \neq 0$; still, the Js in the denominator of the factor to the left get killed because $J^2 = 0$. Thus we have,

$$de^\Lambda = \lambda FJ(\lambda u_\lambda + 1)\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda^2 F u_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F^2 J\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F$$
$$-\lambda^2 F u_\lambda J\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \tag{4.8}$$

where the third term comes from the $\ldots - 1)J$ piece. Note the cancellation. We get,

$$\begin{aligned}
de^\Lambda &= \lambda FJ\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F + \lambda^2 F u_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F^2 J\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\
&= \left(1 + \lambda F u_\lambda \frac{1}{1-\lambda B\frac{F^2-1}{K}J}F\right)\lambda FJ\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F \\
&= e^\Lambda \lambda FJ\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F
\end{aligned} \tag{4.9}$$

thus we have a pure gauge expression for the marginal solution.

To further emphasize the duality with the superstring, note that for the pure gauge problem the role of the eta zero mode is played by the lightcone derivative:

$$\bar{d} \sim \frac{d}{dx^+} \tag{4.10}$$

In particular we have solved the equation,

$$\frac{d}{dx^+}\left(e^{-\Lambda}de^\Lambda\right) = 0 \tag{4.11}$$

Though there are many pure gauge trajectories generated by FXF, only a trajectory which in addition satisfies this equation will be a well-defined, nontrivial solution once spacetime is compactified.

5. Conclusion

In this note, we have constructed analytic solutions of open superstring field theory describing marginal deformations generated by vertex operators with vanishing operator products. We have not attempted to perform any detailed calculations with these solutions, though such calculations are certainly possible. The really important questions about marginal solutions — such as mapping out the relation between CFT and OSFT marginal parameters, obtaining analytic solutions for vertex operators with singular OPEs, or proving Sen's rolling tachyon conjectures [25] — require more work even for the bosonic string. Hopefully progress will translate directly to the superstring.

For us, the main motivation was the hope that marginal solutions could give us a hint about how to construct the vacuum for the open superstring. Indeed, for the bosonic string the marginal and vacuum solutions are closely related: To get the vacuum solution (up to the ψ_N piece), one simply replaces J with $d(Bc) = cKBc$ and takes the limit $\lambda \to \infty$.[11] Perhaps a similar trick will work for the superstring.

The author would like to thank A. Sen and D. Gross for conversations, and A. Bagchi for early collaboration. The author also thanks Y. Okawa for correspondence which motivated discovery of the unitary analytic solution presented in appendix B. This work was supported in part by the National Science Foundation under Grant No.NSF PHY05-51164 and by the Department of Atomic Energy, Government of India.

A. $\mathcal{B}_0, \mathcal{L}_0$ with split strings

In many analytic computations in OSFT it is useful to invoke the operators $\mathcal{B}_0, \mathcal{L}_0$ and their cousins [1, 4]. To avoid unnecessary transcriptions of notation, it is nice to accommodate these types of operations in the split string formalism.

We begin by defining the fields,

$$\mathcal{L} = (\mathcal{L}_0)_L|I\rangle \qquad \mathcal{L}^* = (\mathcal{L}_0^*)_L|I\rangle \tag{A.1}$$

and their b-ghost counterparts $\mathcal{B}, \mathcal{B}^*$. We can split the operators $\mathcal{L}_0, \mathcal{L}_0^*$ into left/right halves non-anomalously because the corresponding vector fields vanish at the midpoint [4]. The fields $\mathcal{L}, \mathcal{L}^*$ satisfy the familiar special projector algebra,

$$[\mathcal{L}, \mathcal{L}^*] = \mathcal{L} + \mathcal{L}^* \tag{A.2}$$

Following ref. [4] we may define even/odd combinations,

$$\mathcal{L}^+ = \mathcal{L} + \mathcal{L}^* = -K \qquad\qquad \mathcal{L}^- = \mathcal{L} - \mathcal{L}^* \tag{A.3}$$

where K is the field introduced before. Note that we have,

$$\begin{aligned} \mathcal{L}_0 \cdot \Psi &= \mathcal{L}\Psi + \Psi\mathcal{L}^* \\ \mathcal{B}_0 \cdot \Psi &= \mathcal{B}\Psi + (-1)^{\Psi}\Psi\mathcal{B}^* \end{aligned} \tag{A.4}$$

We can use similar formulas to describe the many related operators introduced in ref. [4]

Let us now describe a few convenient facts. Let $J(z)$ be a vertex operator for a state $J(0)|\Omega\rangle$ in Schnabl gauge, and let $J = J(1)|I\rangle$ be its corresponding field. Then,

$$[\mathcal{B}^-, J] = 0 \tag{A.5}$$

where $[,]$ is the graded commutator. A similar result $[\mathcal{L}^-, J] = 0$ holds if $J(0)|\Omega\rangle$ is killed by $\mathcal{L}_0$. We also have the useful formulas,

$$\mathcal{L}F = \frac{1}{2}F\mathcal{L}^- \qquad F\mathcal{L}^* = -\frac{1}{2}\mathcal{L}^-F \qquad [\mathcal{L}^-, \Omega^\gamma] = 2\gamma K\Omega^\gamma \tag{A.6}$$

[11] The λ used here and the λ parameterizing the pure gauge solutions of Schnabl [1] are related by $\lambda(Schnabl) = \frac{\lambda}{\lambda+1}$.

The third equation is a special case of,

$$[\mathcal{L}^-, G(K)] = 2KG'(K) \tag{A.7}$$

with similar formulas involving $\mathcal{B}, \mathcal{B}^*$. Of course, these equations are well-known consequences of the Lie algebra eq. (A.2).

As an application, let us prove the identity,

$$\frac{\mathcal{B}_0}{\mathcal{L}_0} J_1(0)|\Omega\rangle * J_2(0)|\Omega\rangle = (-1)^{J_1} F J_1 B \frac{F^2-1}{K} J_2 F \tag{A.8}$$

where $J_1, J_2(0)|\Omega\rangle$ are killed by $\mathcal{B}_0, \mathcal{L}_0$. This expression occurs when constructing the marginal solution (bosonic or superstring) in Schnabl gauge. The direct approach is to compute $\mathcal{L}_0^{-1}$ on the left hand side in split string notation; the resulting derivation is fairly reminiscent of ref. [15]. Instead, we will multiply this equation by $\mathcal{L}_0$ and prove that both sides are equal. The left hand side gives,

$$\begin{aligned}
\mathcal{B}_0 \cdot F J_1 F^2 J_2 F &= \mathcal{B} F J_1 F^2 J_2 F + (-1)^{J_1+J_2} F J_1 F^2 J_2 F \mathcal{B}^* \\
&= \frac{1}{2}(-1)^{J_1} F J_1 [\mathcal{B}^-, F^2] J_2 F \\
&= (-1)^{J_1} F J_1 B F^2 J_2 F
\end{aligned} \tag{A.9}$$

The right hand side gives,

$$\begin{aligned}
\mathcal{L}_0 \cdot F J_1 B \frac{F^2-1}{K} J_2 F &= \mathcal{L} F J_1 B \frac{F^2-1}{K} J_2 F + F J_1 B \frac{F^2-1}{K} J_2 F \mathcal{L}^* \\
&= \frac{1}{2} F J_1 \left[\mathcal{L}^-, B\frac{F^2-1}{K}\right] J_2 F \\
&= F J_1 B \frac{F^2-1}{K} J_2 F + \frac{1}{2} F J_1 B \left[\mathcal{L}^-, \frac{F^2-1}{K}\right] J_2 F
\end{aligned} \tag{A.10}$$

Focus on the commutator:

$$\begin{aligned}
\left[\mathcal{L}^-, \frac{F^2-1}{K}\right] &= [\mathcal{L}^-, F^2]\frac{1}{K} + (F^2-1)\left[\mathcal{L}^-, \frac{1}{K}\right] \\
&= 2F^2 - 2\frac{F^2-1}{K}
\end{aligned} \tag{A.11}$$

where we used eq. (A.7). This computation is a somewhat formal because the inverse of K is not generally well defined, but it can be checked using the integral representation eq. (2.12). Plugging the commutator back in, the $\frac{F^2-1}{K}$ terms cancel and we are left with,

$$\mathcal{L}_0 \cdot F J_1 B \frac{F^2-1}{K} J_2 F = F J_1 B F^2 J_2 F \tag{A.12}$$

which after multiplying by $(-1)^{J_1}$ establishes the result.

Before concluding, we mention that any state of the form,

$$F J_1 B G_2(K) J_2 \ldots B G_n(K) J_n F \tag{A.13}$$

with $[\mathcal{B}^-, J_i] = 0$, is in Schnabl gauge. The proof follows at once upon noting,

$$[\mathcal{B}^-, BG(K)] = -2B^2G'(K) = 0 \tag{A.14}$$

so the entire expression between the Fs commutes with $\mathcal{B}^-$. This is one way of seeing that the nontrivial part of the group element $e^\Phi - 1$ for the superstring solution is in Schnabl gauge.

B. Unitary e^Φ

The analytic solution eq. (3.23) is very simple, but it has the disadvantage of not satisfying the standard reality condition, i.e. e^Φ is not unitary and Φ is not imaginary. Presumably there is an infinite dimensional array of marginal solutions which do satisfy the reality condition, and some may have analytic descriptions. In this appendix we give one construction which is particularly closely related to our solution eq. (3.23). For a very interesting and completely different solution, we refer the reader to an upcoming paper by Okawa [29].

Our strategy will be to find a finite gauge transformation of g in eq. (3.23) yielding a unitary solution. The transformation is,

$$U = Vg \tag{B.1}$$

where V is some string field of the form,

$$V = 1 + dv \tag{B.2}$$

with v carrying ghost number -1. A little thought reveals a natural candidate for V:

$$V = \frac{1}{\sqrt{gg^\dagger}} \tag{B.3}$$

where $g^\dagger$ is the conjugate of eq. (3.23):

$$g^\dagger = 1 - \lambda F \frac{1}{1 - J\lambda B\frac{F^2-1}{K}} XF \tag{B.4}$$

and we use the Hermitian definition of the square root. Intuitively, this is just taking the original solution and dividing by its "norm." More explicitly, if we define,

$$\begin{aligned} gg^\dagger &= 1 + T \\ T &= \lambda FX\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F - \lambda F\frac{1}{1-J\lambda B\frac{F^2-1}{K}}XF \\ &\qquad -\lambda^2 FX\frac{1}{1-\lambda B\frac{F^2-1}{K}J}F^2\frac{1}{1-J\lambda B\frac{F^2-1}{K}}XF \end{aligned} \tag{B.5}$$

then the required gauge transformation is given by the formal sum,

$$V = \frac{1}{\sqrt{gg^\dagger}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} T^n \tag{B.6}$$

This proposal must be subject to two consistency checks. First, of course, is that the field U is actually unitary. The proof is straightforward:

$$\begin{aligned} UU^\dagger &= \frac{1}{\sqrt{gg^\dagger}}gg^\dagger\frac{1}{\sqrt{gg^\dagger}} = gg^\dagger\frac{1}{\sqrt{gg^\dagger}}\frac{1}{\sqrt{gg^\dagger}} = 1 \\ U^\dagger U &= g^\dagger\frac{1}{\sqrt{gg^\dagger}}\frac{1}{\sqrt{gg^\dagger}}g = g^\dagger(g^\dagger)^{-1}g^{-1}g = 1 \end{aligned} \tag{B.7}$$

The second check is that V is a gauge transformation of the form eq. (B.2). This follows if the field T is BRST exact, $T = du$, since then we can write (for example),

$$V = 1 + d\left(\sum_{n=1}^{\infty}\binom{-1/2}{n}uT^{n-1}\right) \tag{B.8}$$

A little guesswork reveals the following BRST exact expression for T:

$$T = d\left(\lambda^2 FX\frac{1}{1-\lambda B\frac{F^2-1}{K}J}B\frac{F^2-1}{K}XF\right) \tag{B.9}$$

This establishes not only that U is an analytic solution, but (perhaps more importantly) that the simpler expression g is in the same gauge orbit with a solution satisfying the physical reality condition. This leaves no question as to the physical viability of our original analytic solution eq. (3.23).

As usual, the unitary solution U can be defined explicitly in terms of cylinder correlators by expanding eq. (B.1) as a power series in λ. Unfortunately this is somewhat tedious because the implicit dependence on λ in eq. (B.1) is complicated. As an expansion for the imaginary field Φ, the first two orders agree with the Schnabl gauge solution (as they must[12]), while at third order we find:

$$\begin{aligned} \Phi_3 = \frac{1}{2}&\left(FXB\frac{F^2-1}{K}JB\frac{F^2-1}{K}JF + FJB\frac{F^2-1}{K}JB\frac{F^2-1}{K}XF\right) \\ &+\frac{1}{4}\left(FXF^2JB\frac{F^2-1}{K} + FJB\frac{F^2-1}{K}XF^2\right)XF \\ &-\frac{1}{4}FX\left(B\frac{F^2-1}{K}JF^2XF + F^2XB\frac{F^2-1}{K}JF\right) + \frac{1}{3}(FXF)^3 \end{aligned} \tag{B.10}$$

This expression is much simpler than the Schnabl gauge solution at third order, which involves intricate constrained and entangled integrals over moduli separating vertex operator insertions.

References

[1] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [hep-th/0511286].

[12] The reality condition fixes the form of the second order solution uniquely within the K, B, J, X subalgebra.

[2] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory*, *JHEP* **04** (2006) 055 [hep-th/0603159].

[3] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory*, *JHEP* **05** (2006) 006 [hep-th/0603195].

[4] L. Rastelli and B. Zwiebach, *Solving open string field theory with special projectors*, hep-th/0606131.

[5] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum*, *JHEP* **02** (2007) 096 [hep-th/0606142].

[6] H. Fuji, S. Nakayama and H. Suzuki, *Open string amplitudes in various gauges*, *JHEP* **01** (2007) 011 [hep-th/0609047].

[7] Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for tachyon condensation with general projectors*, hep-th/0611110.

[8] T. Erler, *Split string formalism and the closed string vacuum*, *JHEP* **05** (2007) 083 [hep-th/0611200].

[9] T. Erler, *Split string formalism and the closed string vacuum. II*, *JHEP* **05** (2007) 084 [hep-th/0612050].

[10] I. Ellwood, J. Shelton and W. Taylor, *Tadpoles and closed string backgrounds in open string field theory*, *JHEP* **07** (2003) 059 [hep-th/0304259].

[11] N. Berkovits, *SuperPoincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [*Erratum ibid.* **B459** (1996) 439] [hep-th/9503099]; *A new approach to superstring field theory*, *Fortschr. Phys.* **48** (2000) 31 [hep-th/9912121].

[12] A. Sen and B. Zwiebach, *Large marginal deformations in string field theory*, *JHEP* **10** (2000) 009 [hep-th/0007153].

[13] A. Iqbal and A. Naqvi, *On marginal deformations in superstring field theory*, *JHEP* **01** (2001) 040 [hep-th/0008127].

[14] M. Schnabl, *Comments on marginal deformations in open string field theory*, hep-th/0701248.

[15] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for marginal deformations in open string field theory*, hep-th/0701249.

[16] J. Kluson, *Exact solutions in SFT and marginal deformation in BCFT*, *JHEP* **12** (2003) 050 [hep-th/0303199];
T. Takahashi and S. Tanimoto, *Wilson lines and classical solutions in cubic open string field theory*, *Prog. Theor. Phys.* **106** (2001) 863 [hep-th/0107046].

[17] I. Kishimoto and T. Takahashi, *Marginal deformations and classical solutions in open superstring field theory*, *JHEP* **11** (2005) 051 [hep-th/0506240].

[18] K. Ohmori, *Comments on solutions of vacuum superstring field theory*, *JHEP* **04** (2002) 059 [hep-th/0204138]; *On ghost structure of vacuum superstring field theory*, *Nucl. Phys.* **B 648** (2003) 94 [hep-th/0208009].

[19] N. Berkovits, Y. Okawa and B. Zwiebach, *WZW-like action for heterotic string field theory*, *JHEP* **11** (2004) 038 [hep-th/0409018].

[20] D.J. Gross and W. Taylor, *Split string field theory. I*, *JHEP* **08** (2001) 009 [hep-th/0105059]; *Split string field theory. II*, *JHEP* **08** (2001) 010 [hep-th/0106036]; L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *JHEP* **11** (2001) 035 [hep-th/0105058]; I. Bars, *Map of Witten's $\star$ to Moyal's$\star$*, *Phys. Lett.* **B 517** (2001) 436 [hep-th/0106157]; M.R. Douglas, H. Liu, G.W. Moore and B. Zwiebach, *Open string star as a continuous moyal product*, *JHEP* **04** (2002) 022 [hep-th/0202087].

[21] Y. Okawa, *Analytic solutions for marginal deformations in open superstring field theory*, arXiv:0704.0936.

[22] N. Berkovits, *Review of open superstring field theory*, hep-th/0105230.

[23] K. Ohmori, *A review on tachyon condensation in open string field theories*, hep-th/0102085.

[24] P.-J. De Smet, *Tachyon condensation: calculations in string field theory*, hep-th/0109182.

[25] A. Sen, *Rolling tachyon*, *JHEP* **04** (2002) 048 [hep-th/0203211]; *Tachyon matter*, *JHEP* **07** (2002) 065 [hep-th/0203265].

[26] N. Berkovits, *The tachyon potential in open Neveu-Schwarz string field theory*, *JHEP* **04** (2000) 022 [hep-th/0001084].

[27] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, *Nucl. Phys.* **B 587** (2000) 147 [hep-th/0002211].

[28] K. Ohmori, *Level-expansion analysis in NS superstring field theory revisited*, hep-th/0305103.

[29] Y. Okawa, *Real analytic solutions for marginal deformations in open superstring field theory*, arXiv:0704.3612.

JHEP07(2007)050